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THE PENETRATION OF A DECELERATED CHARGED-PARTICLE BEAM INTO A PLASMA

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Part I - An Approximate Treatment (S. V. Yadavalli)

SUMMARY

Assuming a decelerated charged particle beam penetrating a uniform plasma, a relation for the growth rate of electrostatic instabilities is obtained. It is shown that the growth rate in a decelerated beam case is always higher than in a uniform one. An experimental method of determining the electron plasma frequency of the plasma and the decelerating parameters
beta alpha
β and α is also suggested.

I. Introduction

In this report we treat the problem of the interaction of a charged particle beam with a plasma, where the plasma was created originally by the passage of the charged particle beam through a neutral gas. This problem is of importance in connection with the experimental work now in progress at the University of Illinois. Also, mention must be made that some experimental work along similar lines (involving the stability of charged particle beams) is being carried on at the Research Laboratory of Electronics at M.I.T. under the direction of L. D. Smullin.

We will study only one-dimensional situations here with the hope that we might be able to extend the treatment later to include the case of a finite beam with axial deceleration.

II. Theoretical "Model"

Since in our problem the charged particle beam creates its own plasma, one expects the energy of the beam to decrease along, say, the direction of motion of the beam. Such a decrease in the energy of the beam is incorporated here according to two models, by

Case I:

assuming that the charged particle beam is decelerated exponentially, that is, in the non-relativistic regime

$$u(z) \sim u_o e^{-\alpha z} \tag{1}$$

where u_o is the (entrance) velocity of the beam at $z = 0$,

α is some positive constant determinable by experiment, and

z is the pertinent spatial coordinate along which deceleration is experienced.

We assume further that the steady beam current is invariant with z. Also, the plasma thus created is assumed to be uniform, and the plasma properties are independent of z.

Case II:

assuming that the charged particle beam is decelerated according to

$$u(z) \sim u_o \left\{1 + \beta\, e^{-\alpha z}\right\} \tag{2}$$

where $\beta > 0$, and $\alpha > 0$.

Note that according to the model of Case II, the beam has some finite energy at $z = \infty$, because

$$\lim_{z \to 0} u(z) \longrightarrow u_o(1 + \beta), \text{ and}$$

$$\lim_{z \to \infty} u(z) \longrightarrow u_o$$

It will be further assumed in this case, just as in Case I, that the beam current is invariant with z, and that the plasma properties are uniform and independent of z.

III. An Approximate Treatment

In this part of the report we will consider only Case II. In Part II of this report, H. H. C. Chang has treated both the cases in a more rigorous fashion.

Let us now proceed with the discussion of Case II. We assume a decelerating (non-relativistic) electron beam penetrating a plasma.* We assume here that the charged particle beam is of such low density that

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* All transverse variations of quantities will be neglected here.

it does not affect the properties of the "plasma" medium through which it traverses. It is assumed that the beam and the plasma are "well mixed." Let the velocity of the beam be u; then,

$$u(z) = u_o\left[1 + \beta\, e^{-\alpha z}\right] + \hat{v}_b(z,\ t) \qquad (3)$$

where $\hat{v}(z,\ t)$ is the perturbation in velocity. Assuming only electric fields, we may write the force equation in linearized form as

$$\frac{\partial \hat{v}_b}{\partial t} + \left[u_o\left(1 + \beta\, e^{-\alpha z}\right)\right]\left(\nabla \cdot \hat{v}_b + \nabla \cdot \left(u_o\beta\, e^{-\alpha z}\right)\right) = -\eta \mathbf{E} \qquad (4)$$

Assuming that all perturbation quantities vary as $e^{i\omega t\ -\ ikz}$ Eq. (4) may be written in the form

$$i\omega\hat{v}_b + u_o\left(1 + \beta\, e^{-\alpha z}\right)\left(-ik\hat{v}_b - u_o\alpha\beta\, e^{-\alpha z}\right) = -\eta[\mathbf{E}(z) + \hat{\mathbf{E}}(z,\ t)] \qquad (5)$$

where $\mathbf{E}(z)$ is the "equivalent" electric field causing the deceleration, and $\hat{\mathbf{E}}(z,\ t)$ is the perturbation in the electric field.

Then,

$$i\omega\hat{v}_b + u_o\left(1 + \beta\, e^{-\alpha z}\right)\left(-ik\hat{v}_b\right) = -\eta\hat{\mathbf{E}} \qquad (5a)$$

and

$$u_o\left(1 + \beta\, e^{-\alpha z}\right)\left(-u_o\alpha\beta\, e^{-\alpha z}\right) = -\eta\mathbf{E}(z) \qquad (5b)$$

Eq. (5a) is the one of interest to us, which may also be written as

$$\hat{v}_b = \frac{i\eta\hat{\mathbf{E}}}{\left\{\omega - ku_o\left(1 + \beta\, e^{-\alpha z}\right)\right\}} \qquad (6)$$

Using the equation of conservation of charge, $\nabla \cdot \hat{j}_b = -\dfrac{\partial\hat{\rho}_b}{\partial t}$, and the fact that the unperturbed beam current is independent of z, which means that $\rho(z)^{*} = \rho_o\left(1 + \beta\, e^{-\alpha z}\right)^{-1}$, we obtain

$$\hat{\rho}_b = \frac{-k\,\rho_o\,\hat{v}_b\left(1 + \beta\, e^{-\alpha z}\right)^{-1}}{\left\{\omega - ku_o\left(1 + \beta\, e^{-\alpha z}\right)\right\}} \tag{7a}$$

or

$$\hat{\rho}_b = \frac{-i\eta\hat{E}\,k\,\rho_o\left(1 + \beta\, e^{-\alpha z}\right)^{-1}}{\left\{\omega - ku_o\left(1 + \beta\, e^{-\alpha z}\right)\right\}^2} \tag{7b}$$

Eq. (7b) is obtained by substituting Eq. (6) in Eq. (7a). In the one-dimensional treatment here it is legitimate to write $\hat{E} = -\frac{\partial\hat{\Phi}}{\partial z}$,** and obtain

$$\hat{\rho}_b = \frac{k^2\,\eta\hat{\Phi}\rho_o\left(1 + \beta\, e^{-\alpha z}\right)^{-1}}{\left\{\omega - ku_o\left(1 + \beta\, e^{-\alpha z}\right)\right\}^2} \tag{8}$$

The following comments are in order. In the plasma we assume equal numbers of ions and electrons, and as the ions are considerably heavier than the electrons, we assume that only the electrons in the plasma contribute to any current due to their motion. Or,

$$\rho_o^{(p)}\,\hat{v}_p = \hat{j}_p \tag{9}$$

where $\rho_o^{(p)}$ is the unperturbed charge density (of electrons alone) in the plasma.***

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* $\rho(z)$ is the charge density of the beam.

** $\hat{\Phi}$ is essentially an electrostatic potential.

*** This charge, however, is assumed to be neutralized in the plasma by ions. Also, $\hat{v}_p$, $\hat{\rho}_p$, and $\hat{j}_p$ are the perturbed velocity, electron charge density and current in the plasma respectively.

Employing the equation of continuity

$$i\omega\hat{\rho}_p = ik\rho_o^{(p)}\,\hat{v}_p$$

we obtain

$$\hat{\rho}_p = \frac{k\rho_o^{(p)}}{\omega}\,\hat{v}_p \tag{10}$$

From the force equation, we have

$$i\omega\hat{v}_p = -\eta\hat{E} = \eta ik\hat{\Phi} \tag{11}$$

or,

$$\hat{\rho}_p = \frac{k^2\hat{\Phi}\,\eta\rho_o^{(p)}}{\omega^2} \tag{12}$$

One may then write Poisson's equation

$$\nabla^2\hat{\Phi} = -\frac{1}{\epsilon_o}(\hat{\rho}_b + \hat{\rho}_p), \text{ and obtain}$$

after some simple algebra that

$$\left\{\eta\rho_o/\epsilon_o \frac{\left(1+\beta\, e^{-\alpha z}\right)^{-1}}{\left[\omega - ku_o\left(1+\beta\, e^{-\alpha z}\right)\right]^2} + \frac{\eta\rho_o^{(p)}}{\epsilon_o\,\omega^2}\right\} = 1 \tag{13}$$

After a little algebra, we find that

$$k = \frac{\omega}{\zeta u_o} \pm \frac{\omega_b}{u_o\zeta^{3/2}}\cdot\frac{1}{(1-\omega_p^2/\omega^2)} \tag{14}$$

where $\zeta = \left(1+\beta\, e^{-\alpha z}\right)$

$\omega_b^2 = -\eta\rho_o/\epsilon_o$

$\omega_p^2 = -\eta\rho_o^{(p)}/\epsilon_o$

ω_b and ω_p are the "electron" plasma frequencies of the beam and the plasma, respectively.

Eq. (14) is the desired relation for the wave number, k. The growth rate is given by the Im k, and

$$\text{Im } k = \frac{\omega_b}{u_o} \frac{1}{\left(1 + \beta\, e^{-\alpha z}\right)^{3/2}} \cdot \left[\frac{\omega_p^2}{\omega^2} - 1\right]^{-1/2} \tag{15}$$

Note that the growth rate is always higher in the case of a decelerated beam than a uniform beam,* other things being the same.

Finally, it may be added that by making measurements at three different frequencies, ω_1, ω_2, and ω_3 say, the three unknown quantities, ω_p, β, and α, can be determined experimentally.

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* This can be seen readily when one calculates an "average" value of k, $\langle k \rangle$, over some distance z = L (not too large), say where $\langle k \rangle \approx \frac{1}{2} \left(k(o) + k(L) \right)$

Part II - A More Rigorous Approach (H. H. C. Chang)

SUMMARY

The stability of a non-relativistic beam of charged particles injected into a cold plasma is studied by means of the macroscopic equations which have been so useful in analyzing microwave tubes in the small signal approximation. In this approach, the simplifying assumptions are made that all quantities of interest vary only in the z direction and that they consist of a d.c. component and a much smaller a.c. component. In particular, I_o, the d.c. component of the beam current, is assumed to be uniform and infinite in extent in the z direction, while v_o, the d.c. component of the beam velocity, and ρ_o, the d.c. component of the charge density, vary with z but in such a manner that $I_o = \rho_o v_o$ is constant. To justify the neglect of the boundary effects of a beam of finite radius 'a', it is assumed that $k_\perp a$ is much greater than unity, where $k_\perp$ is the transverse propagation vector. For low beam densities, collisions between the beam and the plasma are negligible. The plasma is assumed to be fully ionized and only collisions between the electrons and the ions are taken into account by ν, a velocity-independent collision frequency parameter. Collisions between like particles and thermal effects in the cold plasma are both neglected. In Case I, it is assumed that $v_o = V_o \exp(-\alpha z)$ and in Case II that $v_o = V_o[1 + \beta \exp(-\alpha z)]$. The solution of the dispersion relation is discussed in some detail for the special case of wave propagation in the beam direction. The general case when the propagation vector makes an arbitrary angle with the beam direction is considerably more complicated, and with certain approximations holding, the dispersion equation is shown to be an algebraic equation of sixth

degree in ω, the wave frequency. Extension to the case of a relativistic beam is discussed briefly, and it is indicated that the MHD approximation, upon which the present discussion is based, is invalid for presently attainable beam currents.

In this paper, the stability of a non-relativistic beam of charged particles injected into a cold plasma is studied. The beam is confined to move in the z direction only and is infinite in extent in that direction. It is uniform in the xy plane and its radius is 'a'. Boundary effects are neglected by assuming that $\lambda_{\perp}/a$ is much less than unity, where $\lambda_{\perp}$ is the transverse wavelength [1]. This model has been used with considerable success by tube engineers in analyzing microwave, beam type devices. Invariably a sufficiently strong longitudinal magnetic field is assumed to be present to prevent the spreading of the beam due to space-charge repulsion of the electrons of the beam. To keep the analysis tractable, only small signal phenomena will be discussed. This means that the system is perturbed very slightly from an equilibrium state and that all electrical quantities such as v(z), the beam velocity, I(z), the beam current density, $\rho(z)$, the beam charge density, and the electromagnetic field vectors are the sum of a d.c. term and a much smaller a.c. term. Linearization of the relevant equations is justified if the ratio (a.c. term)/(d.c. term) is always negligible compared to unity. Collisions between the beam and the plasma are proportional to the beam current, and for the low beam current that interests us, such collisions may be neglected. The plasma density is sufficiently high so that electron-ion collisions within the plasma must be taken into account by ν, a velocity-independent collision frequency parameter. Collisions between like particles and thermal effects in the plasma are neglected in this rough calculation. With these simplifying assumptions, we are studying the stability against small displacements of a beam that is penetrating at a non-relativistic velocity v(z) another fully ionized medium, the plasma. This model patently applies to microwave beam devices and certain astrophysical and thermonuclear reactor situations.

I. Non-Relativistic Beam

We begin by writing the pertinent equations governing the dynamics of the beam. Let[1]

$$I(z) = I_o + I_1(z)\exp(j\omega t), \tag{1}$$

$$\rho(z) = \rho_o(z) + \rho_1(z)\exp(j\omega t), \tag{2}$$

$$v(z) = v_o(z) + v_1(z)\exp(j\omega t), \tag{3}$$

with

$$I_o = \rho_o v_o = \text{a constant.} \tag{4}$$

In the above equations, the subscript zero denotes d.c. quantities and the subscript one signifies a.c. quantities, which are always much smaller than the corresponding d.c. quantities. By using Maxwell's equations and the Lorentz force law, it is readily shown that I_1 satisfies the following equation: [2]

$$I_1'' + (2j\omega/v_o + 3v_o'/v_o)I_1' + (2j\omega v_o'/v_o^2 - \omega^2/v_o^2 - eI_o/(\epsilon m v_o^3))I_1 = -j\omega e I_o E/(m v_o^3), \tag{5}$$

where ϵ is the permitivity of free space and $I_1' = dI_1/dz$. [2]

[1] See the Glossary for the definition of the more important symbols used in this report. We will use the MKS system of units.

[2] Eq. (5) seems to differ from Eq. (8-35) of [3]. However, they are actually identical. If in (5) we replace I_1, v_o and E by J_ω, u_o and $E_\omega - jJ_\omega/\epsilon\omega$ respectively, we will obtain Hutter's Eq. (8-35).

To find the general solution to (5) let

$$I_1(z) = Y(z)\exp(-j\omega T) \equiv Y(z)G(z) \equiv YG \tag{6}$$

where

$$T = \int_0^z dz/v_o \quad . \tag{7}$$

Putting (6) and (7) into (5) we get

$$Y'' + 3v_o'Y'/v_o - eI_oY/(\epsilon m v_o^3) = -j\omega e I_o E/(m v_o^3 G) = f(z) \quad . \tag{8}$$

Let us first find the homogeneous solutions of (8). Thus, let $y_1(z)$ and $y_2(z)$ be two linearly independent solutions so that the Wronskian, $W(z)$, defined by (9) is non-zero.

$$W(z) = y_1 y_2' - y_2 y_1' \quad . \tag{9}$$

The particular solution of (8) is then [4]

$$y_p(z) = -y_1(z)\int_0^z f(x)y_2(x)/W(x)\,dx + y_2(z)\int_0^z f(x)y_1(x)/W(x)\,dx \tag{10}$$

Case I. $v_o = V_o\exp(-\alpha z)$

Let us assume that due to collisions and other dissipative mechanisms the beam is being slowed down so that

$$v_o = V_o\exp(-\alpha z) \;, \quad z \geqq 0, \quad \alpha \geqq 0. \tag{11}$$

From (4) it follows that

$$\rho_o = \exp(\alpha z), \; G(z) \doteq \exp(-j\omega z/V_o) \quad . \tag{12}$$

Physically this assumption is meaningful only for small z when the approximation

$$\exp(\pm\alpha z) \doteq 1 \pm \alpha z \tag{13}$$

is a good one. Upon inserting (11) into (8) we find that we must solve the following homogeneous equation for $y_1(z)$ and $y_2(z)$.

$$Y'' - 3\alpha Y' + b \exp(3\alpha z) Y = o, \tag{14}$$

where $$b = -eI_o/(\epsilon m V_o^3) = ne^2/(\epsilon m V_o^2) = \omega_p^2/V_o^2 \tag{15}$$

We note in passing that if $\alpha = 0$, the case of no steady electric field or energy loss, (14) becomes

$$Y'' + \omega_p^2 Y/V_o^2 = 0 \tag{16}$$

This case has been discussed by Smullin in Reference 2. To solve (14) for $\alpha \neq 0$, set

$$Y(z) = F(\exp[\zeta z]) = F(s), \quad s = \exp(\zeta z) . \tag{17}$$

Further, if $\zeta = 3\alpha/2$, (14) becomes

$$\frac{d^2F}{ds^2} - \frac{1}{s}\frac{dF}{ds} + \frac{4bF}{9\alpha^2} = 0, \quad \alpha \neq 0 . \tag{18}$$

In Jahnke-Emde, p. 146, the solution of (18) is given as

$$F(s) = sZ_1(a_o s), \quad a_o = 2\omega_p/(3\alpha V_o) , \tag{19a}$$

where

$$Z_n(x) = AJ_n(x) + BN_n(x) \tag{19b}$$

is the cylindrical function and $J_n(x)$ and $N_n(x)$ are the Bessel and

Neumann functions of order n respectively. For convenience we write the solution of (14) as

$$Y(z) = uZ_1(u) , \tag{20}$$

where

$$u = a_o \exp(3\alpha z/2) . \tag{21}$$

It follows from (19) and (20) that the two linearly independent solutions of (14) are

$$y_1(z) = uJ_1(u) \qquad \text{and} \qquad y_2(z) = uN_1(u) \tag{22}$$

with u given by (21). The non-vanishing Wronskian, (9), is

$$W(z) = 3\alpha u^2/\pi . \tag{23}$$

With v_o given by (11) and u by (21), f(z) of (8) can be written

$$f(z) = -a_1 Eu^2/G, \qquad a_1 = 9\ j\omega\alpha^2 I_o e/(4m\omega_p^2 V_o) \tag{24}$$

Putting (22) and (24) into (10) and assuming that E is a constant independent of z, we get

$$y_p(z) = \frac{\pi a_1 Eu}{3\alpha} [J_1(u)S_2(z) - N_1(u)S_1(z)] \tag{25}$$

where

$$S_1(z) = \int_o^z \frac{\underline{u}J_1(\underline{u})\ d\underline{z}}{G(\underline{z})} , \quad S_2(z) = \int_o^z \frac{\underline{u}N_1(\underline{u})\ d\underline{z}}{G(\underline{z})} \tag{26}$$

It is easily shown that $v_1(z)$ is obtained from $I_1(z)$ in the following way: [2]

$$v_1(z) = \frac{v_o}{I_o}\,(I_1(z) - jv_oI_1'(z)/\omega) \tag{27}$$

$$= -\frac{jv_o^2G}{\omega I_o}\,\frac{dY}{dz}\ .$$

Upon inserting (11) and (20) into (27), we obtain $v_{1_h}(z)$, the homogeneous solution of the differential equation which $v_1(z)$ satisfies.

$$v_{1_h}(z) = -\,a_2\exp(\alpha z)G(z)[AJ_o(u) + BN_o(u)]\ , \tag{28}$$

where

$$a_2 = 2j\omega_p^2/(3\omega I_o\alpha)\ . \tag{29}$$

From (6) and (20), $I_{1_h}(z)$, the homogeneous solution of (5), is

$$I_{1_h}(z) = uG(z)[AJ_1(u) + BN_1(u)]\ . \tag{30}$$

To determine the constants A and B, we define z_1 and z_2 by

$$z = z_1 \quad T = T_1 \quad u = u_1\ , \quad N_1(u_1) = 0, \tag{31}$$

$$z = z_2 \quad T = T_2 \quad u = u_2\ , \quad J_o(u_2) = 0\ . \tag{32}$$

With $I_{1_h}(z_1)$ and $v_{1_h}(z_1)$ known from measurements, it follows that

$$A = I_{1_h}(z_1)/[u_1J_1(u_1)G(z_1)]\ , \tag{33}$$

$$B = -v_{1_h}(z_2)\exp(-\alpha z_2)/[a_2N_o(u_2)G(z_2)]\ . \tag{34}$$

The general solution of (8) is

$$Y(z) = u[AJ_1(u) + BN_1(u)] + y_p(z) , \tag{35}$$

where $y_p(z)$ is given by (25) and $u = a_o \exp(3\alpha z/2)$. From (6) the general solution of (5) is

$$I_1(z) = Y(z)G(z) = u[AJ_1(u) + BN_1(u)]G(z) + g(z)E, \tag{36}$$

where

$$g(z) = \frac{-\pi a_1 uG(z)}{3\alpha} [N_1(u)S_1(z) - J_1(u)S_2(z)] , \tag{37}$$

and $S_1(z)$ and $S_2(z)$ are defined by (26). This concludes our description of the beam, and we will now describe the behavior of the plasma.

Let the cold plasma consist of electrons of mass m and ions of mass M, which are at rest in equilibrium with densities n_o and N_o, respectively. In the absence of an external magnetic field, an electric field, $\bar{E}$, causes a current, $\bar{I}_p$, to flow in the plasma, where

$$j\omega\bar{I}_p = \epsilon\left(\frac{\omega\omega_e^2}{\omega - j\nu} + \omega_i^2\right) \bar{E} = \epsilon\omega_t^2 \bar{E} . \tag{38}$$

$$\omega_e^2 = n_o e^2(m + M)/(\epsilon mM) \qquad \omega_i^2 = (n_o - N_o)^2 e^2/[\epsilon(n_o m + N_o M)] \tag{39}$$

and ν is the velocity-independent collision frequency between ions and electrons.

As is well known, $\bar{E}$ satisfies [6]

$$\bar{\nabla} x[\bar{\nabla} x \bar{E}] + \frac{1}{c^2}\ddot{\bar{E}} = -\frac{1}{\epsilon c^2}[\dot{\bar{I}}_p + \dot{\bar{I}}_b] . \tag{40}$$

In (40), $\bar{I}_b$ is the beam current, and from (36) and the one-dimensional model we are using

$$\bar{I}_b = g(z)E_z\bar{e}_z \quad . \tag{41}$$

For small displacements, the equations governing the dynamics of the beam are linear, and one can Fourier analyze the motion and thus regard an arbitrary displacement as a superposition of waves. From the value of the displacement and of its rate of change at every point in space at a given time, combined with the dispersion relation, $\omega = \omega(k)$, one can then calculate the subsequent behavior of the system.

Let us seek solutions of the form $\exp(j[\omega t - \bar{k} \cdot \bar{r}])$. Formally we may replace $\bar{\nabla}$ by $-j\bar{k}$ and $\frac{\partial}{\partial t}$ by $j\omega$. With these substitutions (40) becomes

$$\left(k^2 - \frac{\omega^2}{c^2} + \frac{\omega_t^2}{c^2}\right)\bar{E} - (\bar{k} \cdot \bar{E})\,\bar{k} + j\omega\bar{I}_b \,/\, (\epsilon c^2) = 0. \tag{42}$$

Orient our coordinate axes so that $\bar{k} = k_x\bar{e}_x + k_z\bar{e}_z$. Putting (41) into (42) we obtain the determinant (43) for the desired dispersion relations.

$$\begin{vmatrix} c^2k_z^2 - \omega^2 + \omega_t^2 & 0 & -c^2k_xk_z \\ 0 & c^2\left(k_x^2 + k_z^2\right) - \omega^2 + \omega_t^2 & 0 \\ -c^2k_xk_z & 0 & c^2k_x^2 - \omega^2 + \omega_t^2 + j\omega g(z)/\epsilon \end{vmatrix} = 0 \tag{43}$$

Eq. (43) can be solved explicitly for the case of wave propagation in the direction of the beam (z) axis, when $k_x = 0$. In this special case we find that

$$\omega^2 = \omega_t^2 + c^2k^2 , \qquad (44)$$

is a double root, and

$$\omega^2 = \omega_t^2 + j\omega g(z)/\epsilon \qquad (45)$$

is a simple root. From (15), (24), and (37), (45) can be written

$$\omega = \omega_t[1 + \frac{3}{4}\pi\alpha u G(z)(N_1(u)S_1(z) - J_1(u)S_2(z))]^{-1/2} . \qquad (46)$$

We are interested in the stability of the beam-plasma system against small disturbances excited at $t = 0$. As we have assumed the time dependence to be of the form $\exp(j\omega t)$ and $k = 2\pi/\lambda$ is assumed to be real and positive, growing waves are indicated by the roots $\omega = \omega(k)$ of the dispersion equation, (43), which have negative imaginary parts. Eq. (44) contains no reference to the beam parameters and relates only to the properties of the plasma. In particular, it contains no beam convection term, $kv_o \doteq kV_o$, and if $\nu = 0$ it has real roots only. If $n_o = N_o$ and $m/M \ll 1$, $\omega_i = 0$ and $\omega_e^2 = \omega_p^2 = n_o e^2/(\epsilon m)$ and (44) is identical with the dispersion relationship for transverse oscillations in a plasma [7]. For both $\nu/|\omega| \ll 1$ and $\nu/|\omega| \gg 1$, it can be verified that the imaginary part of ω is positive. We conclude that in the absence of the beam, the plasma can support electromagnetic waves. This is physically reasonable, for the dissipative effects of collisions in the plasma can only lead to the damping of these waves.

In order to be able to discuss the roots of (46) we must obtain an approximate expression for the denominator of the right hand side valid for small z. Using the following approximations,

$$
\begin{aligned}
u &= a_o \exp(3\alpha z/2) \doteq a_o(1 + 3\alpha z/2) \\
J_1(u) &\doteq J_1(a_o) + \frac{3\alpha z}{2}\left[a_o J_o(a_o) - J_1(a_o)\right] \\
uJ_1(u) &\doteq a_o J_1(a_o) + \frac{3\alpha a_o^2 z}{2} J_o(a_o) \\
T &= \int_o^z dz/v_o = \frac{1}{V_o}\int_o^z \exp(\alpha z)\, dz = \frac{1}{\alpha V_o}(e^{\alpha z} - 1) \doteq \frac{z}{V_o}
\end{aligned}
\tag{47}
$$

a lengthy but elementary calculation shows that (46) can be written approximately as

$$\omega^2\left[1 + \frac{\omega_p^2}{18\omega^2}\left(2(e^{-j\beta z} - 1) + j\beta z(3e^{-j\beta z} - 2)\right)\right] = \omega_i^2 + \frac{\omega_e^2}{1 - j\nu/\omega} \tag{48}$$

where $\beta = \omega/V_o$ and $\omega_p^2 = ne^2/\epsilon m$. For further information about the roots of (48), we require the following approximation to be valid:

$$\exp(-j\beta z) \doteq 1 - j\beta z \tag{48'}$$

Substituting (48') into (48), the dispersion equation becomes

$$\omega^3 - (j\nu + j\omega_p^2 z/(18V_o))\omega^2 - [\omega_e^2 + \omega_i^2 + z\omega_p^2\nu/(18V_o)]\omega + j\nu\omega_i^2 = 0 \tag{49}$$

In the absence of collisions $\nu = 0$, and (49) has solutions $\omega = 0$ and $\omega = \omega_o$ where ω_o satisfies

$$\omega_o^2 - jz\omega_p^2\,\omega_o/(18V_o) - \omega_e^2 - \omega_i^2 = 0\ . \tag{50}$$

For small z, $\omega_o^2 \doteq \omega_e^2 + \omega_i^2$. When $\nu/\omega_o \ll 1$, we can write

$$\omega = \omega_o (1 + \xi/\omega_o), \; \xi/\omega_o \ll 1 .$$

Upon substituting this into (49) and using (50) we learn that

$$\omega \doteq \omega_o + \frac{j\nu\omega_e^2}{2\omega_o^2} . \tag{51}$$

The last result means that for wave propagation in the direction of the beam axis and when $\nu/\omega_o \doteq \nu/\sqrt{(\omega_e^2 + \omega_i^2)}$ is negligible compared to unity, the beam-plasma system is stable against small perturbations.

In a fully ionized plasma, three types of interactions between the charged particles are occurring: electron-electron, electron-ion, and ion-ion interactions. We have assumed for simplicity that the electron-ion interactions are dominant and characterized them by a velocity independent collision-parameter ν. The collision frequency of the components of a plasma is a complicated subject which requires careful discussion. For our purposes, we follow Delcroix's book, **INTRODUCTION TO THE THEORY OF IONIZED GASES**, where on page 111, we learn that

$$\nu = 4\pi n w_1 (p_o^2)_{av} \log[h/p_{o_{av}}] \tag{52}$$

This formula shows that ν can take on a large range of values by varying n, the number density, and/or T, the temperature, but in such a way that we are always dealing with a non-relativistic, classical plasma. After specifying ω_p, V_o and the domain of z, it can be shown that by varying ν, (49) can have complex roots with negative imaginary parts. In other words, for certain values of ν, the beam-plasma system is unstable.

In the general case when $k^2 = k_x^2 + k_x^2$, (43) becomes

$$(\omega^2 - \omega_t^2 - c^2k^2)[(1 + P)(\omega^2 - \omega_t^2)(\omega^2 - \omega_t^2 - c^2k^2) - c^2k_x^2(\omega^2 - \omega_t^2)] = 0,$$

or

$$(\omega^2 - \omega_t^2 - c^2k^2)[(1 + P)\omega^4 - (2\omega_t^2 + c^2k^2 + P[\omega_t^2 + c^2k_z^2])\omega^2 +$$

$$\omega_t^2(\omega_t^2 + c^2k^2)] = (\omega^2 - \omega_t^2 - c^2k^2)F(\omega) = 0 . \qquad (53)$$

where

$$P(z) = P = \frac{3}{4} \pi\alpha u G[N_1(u)S_1(z) - J_1(u)S_2(z)] . \qquad (54)$$

Note that when $k = k_z$, (53) yields the earlier results, (44) and (45), the case of wave propagation in the direction of the beam axis, as is required. Recalling (12) and the definition of P, $F(\omega) = 0$ is a transcendental equation in ω. The problem is to determine the nature of its roots. Using (38), the explicit form of $F(\omega) = 0$ is

$$F(\omega) = (1 + P)\omega^6 - 2j\nu(1 + P)\omega^5 - \Big\{[1 + P]\nu^2 + [2 + P](\omega_e^2 + \omega_i^2)$$

$$+ c^2k^2[1 + P\cos^2\theta]\Big\}\omega^4 + j\nu\Big\{[2 + P](\omega_e^2 + 2\omega_i^2) + 2c^2k^2(1 + P\cos^2\theta)\Big\}\omega^3$$

$$+ \Big\{\omega_i^2\nu^2(2 + P) + \omega_e^4 + 2\omega_e^2\omega_i^2 + \omega_i^4 + c^2k^2\nu^2(1 + P\cos^2\theta)$$

$$+ c^2k^2(\omega_e^2 + \omega_i^2)\Big\}\omega^2 - j\nu(2\omega_e^2\omega_i^2 + 2\omega_i^4 + c^2k^2[\omega_e^2 + 2\omega_i^2])\omega$$

$$- c^2k^2\omega_i^2\nu^2 - \nu^2\omega_i^4 = 0. \qquad (55)$$

If $\nu = 0$, $\omega_t^2 = \omega_{t_o}^2 = \omega_e^2 + \omega_i^2$ and (55) becomes

$$F(\omega) = \omega^2[(1 + P)\omega^4 - ([2 + P]\omega_{t_o}^2 + c^2k^2[1 + P\cos^2\theta])\omega^2 + \omega_{t_o}^2(\omega_{t_o}^2 + c^2k^2)] = 0$$

$$(56)$$

Using (6), (19), (21), (26), (47), and (54), P(z) can be approximated by

$$P(z) \doteq \frac{\omega_p^2}{18\omega^2}\left[2(\exp[-j\omega z/V_o] - 1) + \frac{j\omega z}{V_o}(3\exp[-j\omega z/V_o] - 2)\right] \qquad (57)$$

For sufficiently small z we may use the approximation

$$\exp(-j\omega/V_o) \doteq 1 - j\omega z/V_o \qquad (58)$$

to simplify (57) to read

$$P(z) \doteq -\frac{j\omega_p^2 z}{18\omega V_o} = -\frac{p}{\omega}, \quad p = j\omega_p^2 z/18V_o \qquad (59)$$

Putting (59) into (56) we learn that the dispersion equation simplifies to

$$\omega^2[\omega^4 - p\omega^3 - (2\omega_{t_o}^2 + c^2k^2)\omega^2 + (p\omega_{t_o}^2 + pc^2k^2\cos^2\theta)\omega + \omega_{t_o}^2(\omega_{t_o}^2 + c^2k^2)] = 0 \qquad (60)$$

According to a celebrated theorem in the theory of equations due to Euler and Gauss, "Every polynomial p(z) of positive degree with complex coefficients has a complex root." [8]. Eq. (60) fulfills the conditions of this theorem, and we can conclude that the fourth degree polynomial in the square bracket has a complex root. The solution of a quartic is a straightforward, albeit laborious process. On page 254 of Reference 2, the standard recipe for solving quartics is given. It is patent that by varying $k = 2\pi/\lambda$ and/or $p = j\omega_p^2 z/18V_o$, (60) can be made to possess complex roots $\omega = \omega_r + j\omega_i$ where $\omega_i < 0$. In varying p, however, it must be remembered that z must be sufficiently small so that the series expansions of (47) and (58) are valid.

A sensible procedure to follow in our search for the nature of the roots of (56) is to first specify α, ω_p, and V_o. The expansion

$$f(a_o + z) \doteq f(a_o) + f'(a_o)z \tag{61a}$$

is arbitrarily defined as a good approximation if

$$\left|\frac{f'(a_o)z}{f(a_o)}\right| \leqq \frac{1}{1000} \tag{61b}$$

Let z_1, z_2, z_3, and z_4 be the maximum values of z which will make the expansions of (47) valid. Let z_o be the smallest of z_1, z_2, z_3, and z_4. In (60) we then allow z to vary between z = 0 and $z = z_o$. Upon specifying θ, the angle between $\overline{k}$ and the z axis and 'a' the beam radius, we require that

$$k \sin \theta \ll a \tag{62}$$

and investigate if there are any complex roots of (60), satisfying (63), whose imaginary parts are negative. If so, the system is unstable. However, these complex roots are sensible only if $|\omega z_o/V_o|$ is small compared to unity. This last requirement follows from our use of the approximate expression $\exp(-j\omega z_o/V_o) \doteq 1 - j\omega z_o/V_o$ to obtain (60).

When $\nu \neq 0$ and with (59) holding, (55) can be written

$$\begin{aligned}
&\omega^6 - (p + 2j\nu)\omega^5 + (2j\nu p - \nu^2 - 2(\omega_e^2 + \omega_i^2) - c^2k^2)\omega^4 + (p[\nu^2 + \omega_e^2 + \omega_i^2 \\
&\quad + c^2k^2\cos^2\theta] + 2j\nu[\omega_e^2 + 2\omega_i^2 + 2c^2k^2])\omega^3 + \{(2\omega_i^2 + c^2k^2)\nu^2 \\
&\quad - j\nu p[\omega_e^2 + 2\omega_i^2 + 2c^2k^2\cos^2\theta] + (\omega_e^2 + \omega_i^2)[c^2k^2 + \omega_e^2 + \omega_i^2]\}\omega^2 \\
&\quad - j\nu(2\omega_i^2\omega_e^2 + 2\omega_i^4 + c^2k^2[\omega_e^2 + 2\omega_i^2] - j\nu\omega_i^2 p - j\nu c^2k^2 p\cos^2\theta)\omega \\
&\quad - \nu^2\omega_i^2(c^2k^2 + \omega_i^2) = 0
\end{aligned} \tag{63}$$

For arbitrary values of the coefficients of ω, (63) can be solved numerically and the nature of the roots displayed. The procedure outlined previously for the case $\nu = 0$ can be followed without any change. Although this procedure is simple in theory, the actual numerical work involved can be quite laborious for arbitrary values of ν/ω and will not be carried out. A good reference on this subject which will make clear the manifold considerations that are involved is Reference 3, Chapters 10 and 14. This concludes our discussion of the case where $v_o = V_o \exp(-\alpha z)$ and we will continue by discussing a different dependence of v_o on z.

Case II. $v_o = V_o(1 + \beta \exp[-\alpha z])$

Assume that

$$v_o = V_o(1 + \beta \exp[-\alpha z]), \quad 0 < \beta \ll 1, \; \alpha > 0 \; . \qquad (64)$$

As we require that $I_o = \rho_o v_o$ = constant to terms of order β, we must put

$$\rho_o \doteq P_o(1 - \beta \exp[-\alpha z]) \; . \qquad (65)$$

$I_1(z)$, the a.c. current, satisfies (5). If we put

$$I_1(z) = \frac{Y(T)}{v_o} \exp(-j\omega T) = \frac{GY(T)}{v_o} \; , \qquad (66)$$

where

$$T = \int_o^z \frac{dz}{v_o} \; , \quad \text{(5) becomes}$$

$$\frac{d^2Y}{dT^2} - \left[\frac{1}{v_o}\frac{d^2v_o}{dT^2} + \frac{eI_o}{mv_o}\right] Y = \frac{-j e\omega I_o E}{mG} = f_1(T) \qquad (67)$$

Neglecting β compared to unity and with (13) holding, (67) can be approximated by

$$\frac{d^2Y}{dT^2} - (a_3\exp(-\alpha V_o T) + \omega_p^2)Y = f_1(T) \;, \tag{68}$$

where

$$a_3 = \beta\alpha^2 V_o^2 \exp\beta \cong \beta\alpha^2 V_o^2 \;, \quad \omega_p^2 = -eI_o/(\epsilon m V_o) \tag{69}$$

As before, let us first find the homogeneous solutions of (68). To this end, let

$$Y(T) = F(\exp[-\zeta T]) = F(s), \quad s = \exp[-\zeta T] \;, \quad \zeta = \alpha V_o \tag{70}$$

It then turns out that F(s) satisfies

$$s^2 \frac{d^2F}{ds^2} + s\frac{dF}{ds} - \frac{1}{\zeta^2}(a_3 s + \omega_p^2)F = 0. \tag{71}$$

According to page 146 of Jahnke-Emde, solution of (71) is

$$F(s) = AJ_n(2\sqrt{\beta s}\,) + BN_n(2\sqrt{\beta s}\,), \quad n = 2\omega_p/\zeta \tag{72}$$

The Wronskian, W(T) as defined by (9) of $y_1(T) = J_n(2\sqrt{\beta s}\,)$ and $y_2(T) = N_n(2\sqrt{\beta s}\,)$ is

$$W(T) = -\frac{\zeta}{\pi} \tag{73}$$

The particular solution of (68) is, according to (10),

$$y_p(T) = \frac{-j\pi\alpha\omega_p^2\epsilon E}{\alpha}\left[N_n(\Theta)\int_o^T J_n(\underline{\Theta})\exp(j\omega\underline{T})\,d\underline{T} - J_n(\Theta)\int_o^T N_n(\underline{\Theta})\,\exp(j\omega\underline{T})\right]d\underline{T} \equiv \frac{-j\pi\alpha\omega_p^2\epsilon E}{\alpha}\,W(\Theta) \tag{74}$$

where $\Theta = 2\sqrt{\beta s}$.

The complete solution of (5), with (64) holding, is

$$I_1(z) = \frac{G}{v_o}\left[AJ_n(\Theta) + BN_n(\Theta)\right] + g_1(z)E,$$

where

$$g_1(z) = \frac{-j\pi a\omega_p^2 \epsilon G}{\alpha v_o} W(\Theta) \tag{75}$$

The homogeneous solution to the equation which $v_1(z)$ satisfies will be designated by $v_{1_h}(z)$, as before. It follows from (27) and (75) that

$$v_{1_h}(z) = \frac{j\alpha v_o^2 \Theta G}{2\omega I_o}\left[AJ_n'(\Theta) + BN_n'(\Theta)\right], \tag{76}$$

where $J_n'(\Theta) = dJ_n(\Theta)/d\Theta$. $I_{1_h}(z)$, the homogeneous solution of (5), is

$$I_{1_h}(z) = \frac{G}{v_o}\left[AJ_n(\Theta) + BN_n(\Theta)\right]. \tag{77}$$

To determine the constants A and B, we define z_1 and z_2 by

$$z = z_1, \quad T = T_1, \quad \Theta = \Theta_1, \quad N_n(\Theta_1) = 0$$

$$z = z_2, \quad T = T_2, \quad \Theta = \Theta_2, \quad J_n'(\Theta_2) = 0$$

With $I_{1_h}(z_1)$ and $v_{1_h}(z_1)$ known from measurements, it follows that

$$A = I_{1_h}(z_1)V_o/[G(T_1)J_n(\Theta_1)]$$

$$B = 2\omega I_o v_{1_h}(z)/[\Theta_2 V_o^2 G(T_2)]$$

The interaction between the beam and the plasma is accounted for exactly as in the previous case. Thus, the analysis following Eq. (37) is applicable with slight changes in notation. Eq. (41) becomes

$$\overline{I}_b = g_1(z)E_z\overline{e}_z , \tag{79}$$

and (43) still holds with $g_1(z)$ replacing $g(z)$. Eq. (44) is unchanged and (46) becomes

$$\omega = \frac{\omega_t}{[1 + \pi\omega_p^2 G(z)W(\Theta)/(\alpha V_o)]^{1/2}} \tag{80}$$

In the general case, when $k^2 = k_x^2 + k_z^2$, we get an equation of the form given by (55) except here we must replace $P(z)$ by $P_1(z)$ where

$$P_1(z) = \pi\omega_p^2 G(z)W(\Theta)/(\alpha V_o) \tag{81}$$

For sufficiently small z, α and β we can obtain an approximate expression for $W(\Theta)$ as defined by (74). Thus by means of series expansions similar to (47) we learn that $W(\Theta) \doteq -\alpha V_o T^2/(2\pi)$ and

$$P_1(z) \doteq \frac{-\omega_p^2}{2V_o^2} (1 + 2\beta)(1 + [1 + \beta] \frac{j\omega z}{V_o})z^2 \tag{82}$$

When $j\omega z(1 + \beta)/V_o$ is much less than unity, we approximate $P_1(z)$ by

$$P_1(z) \doteq \frac{-(1 + 2\beta)z^2\omega_p^2}{2 V_o^2} \tag{83}$$

Putting $P \longrightarrow P_1(z)$ in (55) we get the sixth degree dispersion equation which must be solved in order to predict if the beam-plasma system is stable or not. As in Case I, this sixth degree equation can be solved numerically after specifying the parameters which relate to the properties of the beam and the plasma.

II. Relativistic Beam

The rest mass of an electron is about one-half MEV. The kinetic energy of an electron in rest mass units is $\gamma - 1$, and when $\gamma - 1$ is much greater than unity, the previous considerations must be modified because the neglected relativistic corrections are then important. For a relativistic beam, examination of the equations upon which our crude one-dimensional model is based shows that the linearized equation of continuity does not change and there is no change in Poisson's equation unless $\omega/k = c$. [9]. To the first approximation, all we need to do to include relativistic effects is to replace m by $m_o\gamma^3$ as far as the beam is concerned. A more sophisticated and accurate treatment based on Boltzmann's equation is given in Reference 5.

In truth for extremely energetic beams, say when $\gamma - 1$ is much greater than 100, the situation is much more complicated than just indicated. For one thing, as Budker has shown, for highly relativistic beams, radiation is an important dissipative mechanism [10]. Moreover, if the beam energy exceeds the relevant threshold energies, other elementary particles such as mesons could be produced copiously. Additionally this entire discussion is essentially a magnetohydrodynamic one and that, as Finkelstein and Sturrock [11] have shown for a relativistic beam, the MHD approximation is valid provided the number of electrons per classical electron radius, measured in the rest frame of the rapidly moving electrons, is large compared with unity. In practice, this last requirement is equivalent to the statement that the current of the beam must exceed 17,000 γ amps and this current is much greater than currents so far contemplated in relativistic-stream experiments.

In Reference 5, the stability of a uniform, relativistic beam of particles injected into a plasma is studied using the Boltzmann transport equation. The most important results which were obtained are: (1) The beam is relativistic so that in the equations of motion the relativistic masses γm_o and $\gamma^3 m_o$ appear. This makes the beam "stiff" so that magnetic self-forces, which for non-relativistic pinches lead to kinking, are negligible compared with electrostatic forces. (2) The velocity spread in the beam is allowed to be appreciable so that for short enough wavelengths the velocity structure of the beam is important. The velocity spread in the plasma penetrated is small. (3) As in this report, only electron-ion collisions are considered important and are taken care of via a velocity independent collision frequency parameter. In this important paper, the effects of boundary conditions in a finite beam and the non-uniform structure present when velocity gradients are significant are assumed to be small and consequently negligible.

In Reference 12, two-stream instability in finite, relativistic streams is investigated. The electrostatic mode is singled out for discussion, and a detailed study is made of effects associated with the finite size and non-uniformity of the beam. An eigenvalue equation for E_z, the longitudinal electrical field, is obtained and for a weak-beam it is shown that two-stream instability can exist only if η is real and $\eta \geqq 1$. Here η is a function of ω, v, and the plasma frequency of the beam and is defined by Eq. (29) of this reference.

Finally we note that in some ways a study of the instabilities of a relativistic beam is simpler than that of a non-relativistic beam. In using a one-dimensional model to describe a non-relativistic beam it is imperative that a large longitudinal d.c. magnetic field be present to inhibit transverse motion of the electrons and thus keep the beam from spreading. An extreme relativistic beam is pinched by its own magnetic field and when the Bennett condition [11] is satisfied, self-magnetic effects are sufficient to hold the beam together against its random transverse kinetic energy and multiple scattering effects without an externally applied longitudinal magnetic field. Obviously the complexity of the analysis is strongly dependent on the parametric choices made as to whether the skin depth is large or small compared to the beam radius and the magnitude of the ratio $|\nu/\omega|$. The various instabilities are all means of transferring energy from the beam to the plasma substratum, and they divide naturally into those which do and which do not depend on edge effects--whether the beam has sharp edges or diffuse edges. A good reference on the stability of a relativistic beam is Reference 13.

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GLOSSARY

a	beam radius
a_o	constant defined by Eq. (19a)
a_1	constant defined by Eq. (24)
a_2	constant defined by Eq. (29)
a_3	constant defined by Eq. (69)
b	$b = \omega_p^2/V_o^2$, Eq. (15)
c	velocity of light
e	magnitude of the charge of the electron, $e = + 1.6 \times 10^{-19}$ coulomb
$\bar{e}_z$	unit vector in the z direction, the beam direction
$\bar{E}$	intensity of the electric field vector
F	function of z defined by Eq. (17)
$g(z) = g$	function of z defined by Eq. (37)
$G(z) = G$	$G = \exp(-j\omega T)$
I	beam current, $I(z) = I_o + I_1(z) \exp(j\omega t)$
I_o	d.c. component of beam current
$I_1(z) = I_1$	a.c. component of beam current
I_1'	$I_1' = dI_1/dz$
j	imaginary unit, $j^2 = -1$
$J_n(z)$	Bessel function of order n
$\bar{k}$	propagation vector
$k_{\parallel}$	z component of $\bar{k}$, $k_{\parallel} = k \cos \Theta$
$k_{\perp}$	normal component of $\bar{k}$, $k_{\perp} = k \sin \Theta$
m	electron mass
M	ion mass

n_o	number density of electrons of the plasma
N_o	number density of ions of the plasma
$N_n(z)$	Neumann function of order n
p	quantity defined by Eq. (59)
$P(z) = p$	quantity defined by Eq. (54)
s	variable defined by Eq. (17), $s = \exp(\zeta z)$. Also $s = \exp(-\zeta T)$
S_1, S_2	quantities defined by Eq. (26)
t	time
T	$T = \int_o^z dz/v_o$
u	variable defined by Eq. (21), $u = a_o \exp(3\alpha z/2)$
$\overline{v}(z) = \overline{v}$	velocity of the beam, $\overline{v} = v\,\overline{e}_z$, $v = v_o + v_1(z)\exp(j\omega t)$
$v_o(z) = v_o$	d.c. component of v
$v_1(z) = v_1$	a.c. component of v
V_o	magnitude of v_o in Case I at $z = 0$; quantity defined in Eq. (64) related to the magnitude of v_o
W(z)	Wronskian, Eq. (9)
z	the independent variable
Z_n	the cylindrical function, Eq. (19b)
α	exponential decrement factor, Eq. (11) and Eq. (64)
β	in Case I, $\beta = \omega/V_o$. β is also used as in Eq. (64)
γ	$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$
ϵ	permitivity of free space
ζ	$\zeta = \alpha V_o$
θ	angle between $\overline{k}$ and $\overline{e}_z$

λ	wavelength, $k = 2\pi/\lambda$
$\lambda_\perp$	$\lambda_\perp = \lambda \sin \theta$
$\lambda_\parallel$	$\lambda_\parallel = \lambda \cos \theta$
ν	collision frequency between electrons and ions, Eq. (52)
$\rho(z) = \rho$	electric charge density, $\rho = \rho_o + \rho_1$
ρ_o	d.c. component of ρ
ω	wave frequency
ω_e	quantity defined by Eq. (39)
ω_i	quantity defined by Eq. (39)
ω_p	plasma frequency, Eq. (15)
ω_t	quantity defined by Eq. (39)
ω_o	$\omega_o^2 \doteq \omega_e^2 + \omega_i^2$